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Effect of the injection rate on the multifractality of the flow distribution in river network models

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Abstract. The multifractal structure of the flow (channel discharge) distribution is investigated in river network models which are extended versions of Scheidegger's river network model. Two models are proposed: in the first model the constant injection rate in Scheidegger's model is replaced by an uncorrelated random variable and in the second model the injection rate is given by that of a power law L^{β} where L is the length in the downstream direction. The effect of the injection rate on the multifractality of the flow distribution is studied by Monte Carlo simulation. It is shown that the partition function $Z(q) \equiv \Sigma_i I_i^q$ scales as $Z(q) \approx L^{\zeta(q)}$ where I_i is the flow (channel discharge) of water passing over the bond i within the river network and the summation ranges over all bonds. In the first model, for a large L, the multifractal structure of the flow distribution agrees with that of Scheidegger's model. In the second model, it is found that the power law injection rate has an important effect on the multifractality.

1. Introduction

Recently, there has been increasing interest in fractal structures of growth processes such as diffusion-limited aggregation (DLA), ballistic deposition and river networks. The DLA model presents a prototype of the pattern formation of diffuse systems including electrodeposition, crystal growth, viscous fingering and bacterial colonies [1–8]. The ballistic deposition model provides a basis for understanding the deposition processes used to prepare a wide variety of thin-film devices [9]. Branched river networks are among nature's most common patterns, spontaneously producing fractal structure [10].

The multifractal properties of the DLA and the random resistor network have recently attracted considerable attention [11–14]. It has become clear that the DLA aggregate cannot be fully characterized by its fractal dimensionality. In order to characterize the aggregate further, it is necessary to derive the multifractal structure of the growth probability distribution. From the multifractality, one can obtain detailed information on the capability of each perimeter site to grow and, therefore, more information on the surface structure [11, 13]. Also, it has been found that the electric current distribution on the percolation cluster shows multifractal structure [15]. It has been shown that electrical properties of self-similar resistor networks should be characterized by an infinite set of exponents.

Rivers have been studied extensively by a diverse range of researchers with a variety of techniques and goals [16, 17]. Geomorphologists have found scaling relationships among various combinations of basin statistics from field data, such as drainage density and branching ratios. Hydrologists have likewise extracted power laws for channel

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parameters such as width, depth and velocity as functions of total channel discharge. Some investigators have constructed models for the evolution of an entire drainage network [18–20]. Scheidegger's model is the simplest model which reveals the essential features of river formation. It is known that the size distribution of rivers in Scheidegger's model satisfies the power law and the river pattern shows the self-affine fractal structure from theoretical methods and computer simulation [21–23]. Very recently, it was found that the flow (channel discharge) distribution in Scheidegger's river network shows a typical multifractal structure [24]. It was shown that the partition function $Z(q) \equiv \Sigma_i I_i^q$ scales as $Z(q) \approx L^{\zeta(q)}$ where I_i is the flow of water passing over the bond i within the river network, the summation ranges over all bonds and L is the size of river network. It was found that the river width distribution also shows multifractality if the width w of a river scales as $w \approx I^\beta$.

In this paper, we extend Scheidegger's river network model to take into account the variable injection rate. We present two extended versions of Scheidegger's river model. In the first model, the constant injection rate in Scheidegger's model is replaced by an uncorrelated random variable. The strength of the random injection rate represents the density of stationary falling rain. The first model can take into account the effect of the quantity of rain on the flow distribution in the river network. In the second model, we replace the constant injection rate by the injection rate of the power law L^β where L is the length in the downstream direction. The second model represents the river formation process with increasing falling rain. The quantity of falling rain increases according to the river's flow in the downstream direction. We investigate the effect of the injection rate on the multifractality of the flow distribution by the use of Monte Carlo simulation. The variable injection rate does not change the geometrical properties of the river network. However, it has an important effect on the flow distribution.

In the case of a constant injection rate, the flow (channel discharge) coincides with the drainage basin area. In real river networks, the flow distribution is not consistent with the distribution of the drainage basin area. Generally, the rainfall upstream is different from that downstream. The local rainfall changes as it goes downstream. The simple river network model with this property has been little investigated. The second model is presented to take into account a variable rainfall. We are interested in the scaling behaviour of a river network. Therefore, we consider the second model with the injection rate of the power law L^β.

The paper is organized as follows. In section 2 we describe the extended models of Scheidegger's river network. In section 3 we present the simulation results. We derive the scaling of the partition function and the multifractality of the flow distribution. It is shown that in the first model the multifractal structure of the flow distribution agrees with that of Scheidegger's model and in the second model the power law injection rate has an important effect on the multifractality. In section 4 we give a summary.

2. Models

First we introduce Scheidegger's river network model [21]. We extend Scheidegger's model to take into account a variable injection rate. In Scheidegger's model, rain is assumed to be stationary and to fall uniformly on oblique square lattices. One unit of water is injected into each lattice site per unit time. Then, fallen raindrops run down the slope. When two raindrops collide with each other, they join and make one drop, which runs down just as before the collision. Any splitting of the flow is prohibited. Flows are allowed to go right or left but only in the downstream direction. As the result, the rivers do not contain any loops. All branches are directed upstream. One of the most important quantities in this system is the distribution of flow rates in the river network. The rate of flow in the river network is proportional to the area of its drainage basin. The flow rate I_i on the bond i is defined as the amount of flowing water through the bond i per unit time. Each bond of the river network can be characterized by the flow rate of water. If the bond (or site) is labelled by the position (m, n) where m indicates the downstream direction, the flow rate satisfies the equation

$$I(m+1, n) = w(m, n)I(m, n) + [1 - w(m, n+1)]I(m, n+1) + 1 \qquad (1)$$

where $w(m, n)$ denotes the realization of the flow direction at the site (m, n) which is equal to 1 when the flow at the site (m, n) goes down right and 0 when the flow goes down left and $w(m, n)$ is given by

$$w(m, n) = \begin{cases} 1 & \text{probability } \frac{1}{2} \\ 0 & \text{probability } \frac{1}{2}. \end{cases} \qquad (2)$$

We extend Scheidegger's model to take into account the variable injection rate. We present two versions of Scheidegger's model. In the first model, the constant injection rate in equation (1) is replaced by an uncorrelated random variable. The strength of the random injection rate represents the density of stationary falling rain. The first model can take into account the effect of the quantity of rain on the flow distribution. The first model is given by

$$I(m+1, n) = w(m, n)I(m, n) + [1 - w(m, n+1)]I(m, n+1) + \eta(m+1, n) \qquad (3)$$

where the injection rate $\eta(m, n)$ represents white noise with a mean $\langle\eta\rangle = c$ (c: a constant < 1) and $\langle\eta(m', n')\eta(m, n)\rangle = c^2\delta_{m'm}\delta_{n',n}$.

In the second model, we consider the river formation process with increasing falling rain. The quantity of falling rain increases according to the river's flow in the downstream direction. We replace the constant injection rate of equation (1) by the injection rate of the power law L^β where L is the length in the downstream direction. The second model is given by

$$I(m+1, n) = w(m, n)I(m, n) + [1 - w(m, n+1)]I(m, n+1) + m^\beta. \qquad (4)$$

The variable injection rate does not change the geometrical properties of the river network. However, it has an important effect on the flow distribution. In the second model, the flow rate on the river network is not consistent with the area of its drainage basin. We study the effect of the injection rate on the multifractality of the flow distribution.

3. Simulation results

We perform the computer simulation of the first and second models for the square lattice 300×300. The flow rate $I(m, n)$ on each bond is calculated under a periodic lateral boundary condition. In the first model, the injection of raindrops is added on each site independently with the probability c. As an illustration, figure 1 shows the patterns obtained by simulation of equation (3). Figures 1(a) and 1(b) are obtained

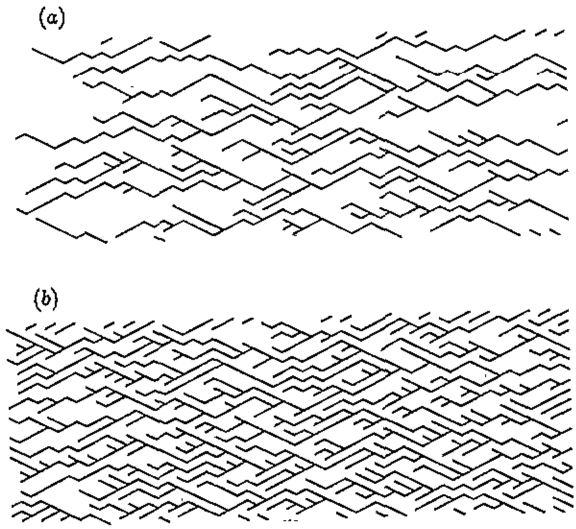

Figure 1. Typical river network patterns generated by the first extended Scheidegger's river model. These runs were done on a 20×60 square lattice under a periodic lateral boundary condition for an illustration. (a) The river pattern obtained under the injection rate of probability $c=0.1$. (b) The pattern with $c=0.5$.

under $c=0.1$ and $c=0.5$, respectively, for size 20×60. We investigate the scaling behaviour of the river pattern. We define the partition function $Z(q)$ of the moments of flow rate as

$$Z(q)\equiv\sum_i I_i^q \tag{5}$$

where the summation ranges over all bonds in the river network. Figure 2 shows the log–log plot of the moments against the size L in the downstream direction for $c=0.5$. It is confirmed that, for large L, the partition function scales with size L as

$$Z(q)\approx L^{\zeta(q)}. \tag{6}$$

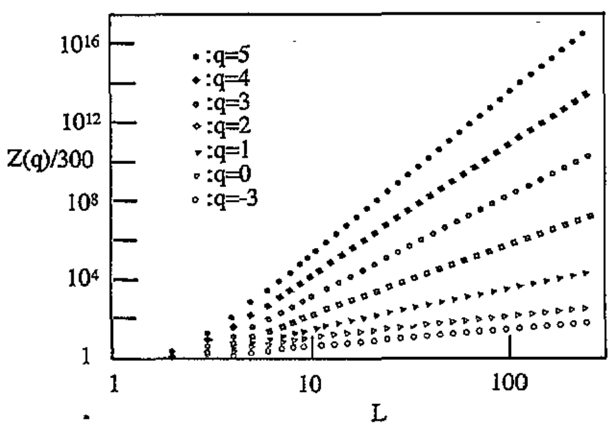

Figure 2. The log–log plot of the moments $Z(q)$ defined by equation (5) against the size L for $c=0.5$ showing the scaling behaviour (6) in the first model.

Figure 3 shows the $Z(3)$ behaviour against L for $c=0.1$, 0.3, 0.5, 0.7, 0.9 and 1. For L larger than 10, the slopes of $Z(3)$ are a constant independent of c. The case $c=1.0$ corresponds to the original Scheidegger's model. The multiscaling exponent $\zeta(q)$ is consistent with that of Scheidegger's model [24]. In the first model with the random injection rate, the river pattern in figure 1 is apparently different from that of Scheidegger's river. However, the scaling properties agree with those of Scheidegger's river. The exponent $\zeta(0)$ is equal to 1. The exponent $\zeta(1)$ gives the dimension $d=2$ of the river network. For a sufficiently large q, $\zeta(q)/q$ gives the scaling exponent 1.50 ± 0.02 for the largest flow rate, which is consistent with the exponent of the drainage basin.

In order to characterize the multifractality of flow distribution, it is convenient to normalize the flow rate. The normalized flow rate I_i' on the bond i is defined as

$$I_i' = I_i/Z(1). \tag{7}$$

The normalized partition function $Z'(q)$ is given by

$$Z'(q) = Z(q)/\{Z(1)\}^q. \tag{8}$$

For a sufficiently large L, the partition function scales as

$$Z'(q) \approx L^{-\tau(q)}. \tag{9}$$

With the Legendre transformation of $\tau(q)$, we obtain the f–α spectrum

$$f(q) = q\alpha(q) - \tau(q) \tag{10}$$

where $\alpha(q) = \partial\tau(q)/\partial q$ is the variable conjugate to q. Figure 4 shows the f–α spectra for $c=0.5$ and 1.0. The f–α spectrum of $c=0.5$ agrees with that of Scheidegger's model ($c=1.0$). The maximum value $f(0)$ of $f(\alpha)$ is related to the dimension $d=2$ of the river network: $f(0)=d-1=1$. The maximum value of α gives the minimum fraction of flow rate. The minimum value of α gives the maximum fraction of flow rate. The minimum value $\alpha(\infty)$ is exactly related to the exponent d_f of the drainage basin:

$$\alpha(\infty) = (\partial\tau/\partial q)_{q=\infty} = 2 - d_f. \tag{11}$$

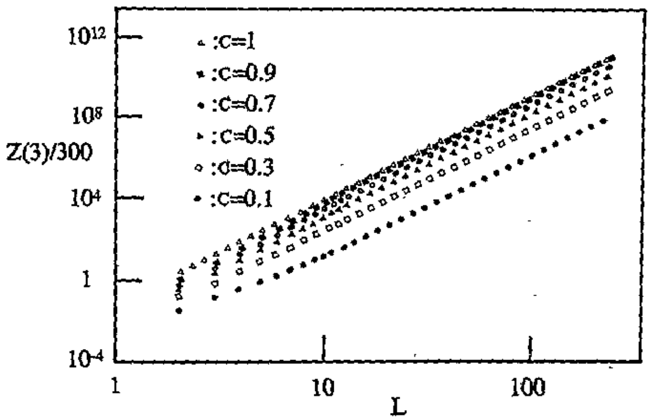

Figure 3. The third-order moment $Z(3)$ behaviour plotted against L for $c=0.1$, 0.3, 0.5, 0.7, 0.9 and 1 in the first model. For L larger than 10, the slopes of $Z(3)$ are a constant independent of the probability c.

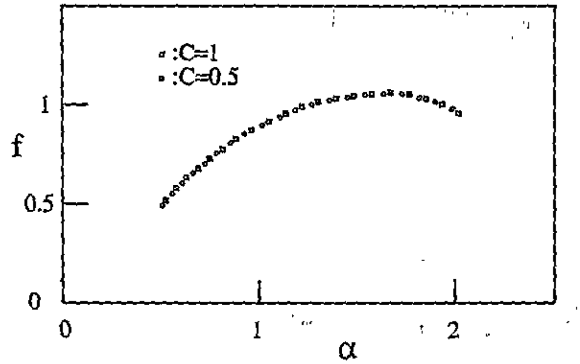

Figure 4. The f–α spectra of the flow distribution for $c=0.5$ and $c=1$ in the first model.

The minimum value $\alpha(\infty)$ obtained from our simulation is given by 0.49 ± 0.03. We obtain the exponent $d_f=1.5\pm0.03$ from equation (11). This value is consistent with the theoretical result of 1.50 [21, 22]. The properties of river networks should be characterized by the infinite set of exponents or the f–α spectrum. The river pattern with a random injection rate is apparently different from that of Scheidegger's river (see figure 1) but the multifractal scaling property is consistent with that of Scheidegger's river.

We now consider the second model described by equation (4). Figure 5 shows the log-log plot of the moments against the size L for $\beta=0.5$. It is confirmed that the partition function scales with size L as $Z(q)\approx L^{\zeta(q)}$. The slope of $Z(3)$ increases with the parameter β. The case of $\beta=0$ is consistent with Scheidegger's model. Figure 6 represents the exponent $\zeta(q)$ plotted against q for various β. For $\beta>0$, the exponent $\zeta(q)$ approaches zero for a sufficiently small negative value of q. For positive values of q, the exponent $\zeta(q)$ deviates from those of Scheidegger's model ($\beta=0$) with increasing β. From equations (7)–(10), we can calculate the f–α spectra. With increasing β, the maximum values of $\alpha(q)$ increase. However, the f–α spectra are consistent with

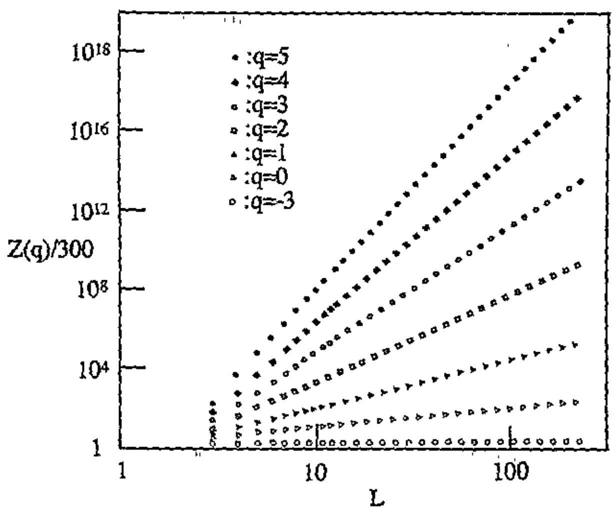

Figure 5. The log-log plot of the moments $Z(q)$ defined by equation (5) against the size L for $\beta=0.5$ for the second model (4).

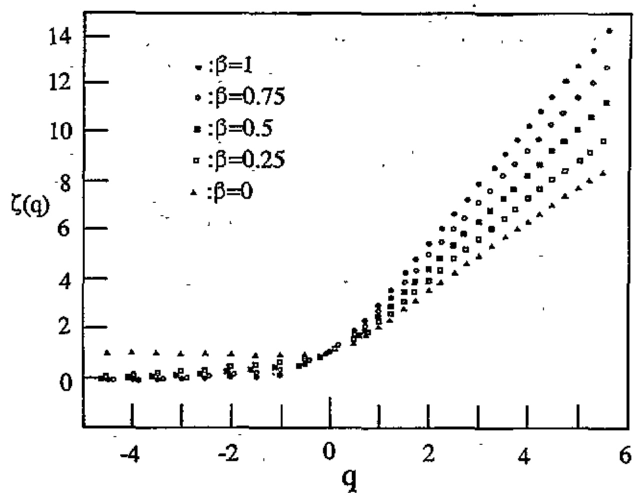

Figure 6. The behaviour of the scaling exponent $\zeta(q)$ against q for various β in the second model.

each other for the region $\alpha < 2$. The minimum value $\alpha(\infty)$ agrees nearly with the value $\alpha(\infty) = 0.5$ of Scheidegger's model. In the first model and Scheidegger's model, the flow rate on the river network is consistent with the area of its drainage basin, but this is not so for the second model. In the second model, the flow rate increases with a larger power than that of Scheidegger's model. The largest flow rate is given by $\zeta(q)/q$ for large q: the slope for large q in figure 6. We find that the power law injection rate has an important effect on the multifractality of the flow distribution.

4. Summary

We have presented two extended versions of Scheidegger's river network model. We investigated the effect of variable injection rates on the flow distribution in the river network. We derived the multifractal structures of the flow distribution in the extended models. In the first model, for a large L, we showed that the multifractal structure of the flow distribution does not change with the quantity of falling rain and agrees with that of Scheidegger's model. We also found that the power law injection rate has an important effect on the multifractal structure of the flow distribution in the second model.

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